ON THE STABILITY OF THE OPTIMAL VALUE AND THE OPTIMAL SET IN OPTIMIZATION PROBLEMS

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Abstract. The paper develops a stability theory for the optimal value and the optimal set mapping of optimization problems posed in a Banach space. The problems considered in this paper have an arbitrary number of inequality constraints involving lower semicontinuous (not necessarily convex) functions and one closed abstract constraint set. The considered perturbations lead to problems of the same type as the nominal one (with the same space of variables and the same number of constraints), where the abstract constraint set can also be perturbed. The spaces of functions involved in the problems (objective and constraints) are equipped with the metric of the uniform convergence on the bounded sets, meanwhile in the space of closed sets we consider, coherently, the Attouch-Wets topology. The paper examines, in a unified way, the lower and upper semicontinuity of the optimal value function, and the closedness, lower and upper semicontinuity (in the sense of Berge) of the optimal set mapping. This paper can be seen as a second part of the stability theory presented in [17], where we studied the stability of the feasible set mapping (completed here with the analysis of the Lipschitz-like property).

1. Introduction

In this paper we consider optimization problems formulated in the form

$$\begin{array}{lll} (\mathrm{P}) & \inf & f(x) \\ & \text{s.t.} & f_t(x) \leq 0, \quad \forall t \in T; \\ & & x \in C, \end{array}$$

where T is an arbitrary (possibly infinite, possibly empty) index set, $\emptyset \neq C \subset X$ is the (abstract) constraint set, the decision space X is a Banach space, and all the involved functions $f, f_t, t \in T$, are extended real-valued, i.e., $f, f_t\colon X \to \mathbb{R} \cup \{+\infty\}$. In this paper we analyze the stability of the optimal value function and the optimal set mapping of (P), say ϑ and $\mathcal{F}^{opt}$, under different possible

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types of perturbations of the data preserving the decision space X and the index set T.

In [17] we studied the effect on the solution set of the constraint system

$$\sigma := \{f_t(x) \leq 0, t \in T;\ x \in C\},$$

also represented by its corresponding data set, $\{f_t, t \in T;\ C\}$, of perturbing any constraint function f_t, $t \in T$, and possibly the constraint set C, under the condition that these perturbations maintain certain properties of the constraints. In particular, we analyzed there the continuity properties in different senses ([4], [36], etc.) of the feasible set mapping associating to each perturbed system its corresponding solution set. Different parametric spaces were considered in [17], such that each one, denoted by $\Theta_\diamond$ (with some subindex) is a given family of systems with the same decision space and index set, satisfying certain particular properties. The main goal of [17] was to study the stability of the *feasible set mapping* $\mathcal{F}\colon \Theta_\diamond \rightrightarrows X$ such that

$$\mathcal{F}(\sigma) \ = \{x \in X : f_t(x) \leq 0,\ \forall t \in T;\ x \in C\}.$$

Many times in this paper we shall use the so-called *marginal function*

$$g := \sup\{f_t, t \in T\},$$

provided that $T \neq \emptyset$. Then we can also write

$$\mathcal{F}(\sigma) \ = \{x \in X : g(x) \leq 0;\ x \in C\}.$$

The parametric space in the present paper is

$$\Pi := \left\{ \pi = (f, \sigma) : \begin{array}{l} f,\ f_t\colon X \to \mathbb{R} \cup \{+\infty\},\ t \in T, \text{ are lsc,} \\ \text{and } C \text{ is closed} \end{array} \right\},$$

where $\sigma = \{f_t, t \in T;\ C\}$, and lsc stands for lower semicontinuous. Consequently, g is lsc too, and it is also upper semicontinuous (usc) whenever f_t is usc for all $t \in T$ and $|T| < \infty$.

The first objective of this paper consists of analyzing the *optimal value function* $\vartheta\colon \Pi \to \mathbb{R} \cup \{\pm\infty\}$ defined as

$$\vartheta(\pi) := \inf\{f(x) : x \in \mathcal{F}(\sigma)\} = \inf f(\mathcal{F}(\sigma)),\ \text{with } \pi = (f, \sigma),$$

under the convention $\vartheta(\pi) = +\infty$ if $\mathcal{F}(\sigma) = \emptyset$ (i.e. if $\sigma \notin \operatorname{dom} \mathcal{F}$). If $\vartheta(\pi) = -\infty$ we say that π is *unbounded* (from an optimization point of view).

The second objective of this paper is the stability analysis of the *optimal set mapping* $\mathcal{F}^{opt}\colon \Pi \rightrightarrows X$, i.e.,

$$\mathcal{F}^{opt}(\pi) := \{x \in \mathcal{F}(\sigma) :\ f(x) = \vartheta(\pi)\},\ \text{with } \pi = (f, \sigma).$$

If $\pi \in \operatorname{dom} \mathcal{F}^{opt}$ (i.e. $\mathcal{F}^{opt}(\pi) \neq \emptyset$) we say that π is (optimally) *solvable.* Obviously, if $\pi = (f, \sigma) \in \operatorname{dom} \mathcal{F}^{opt}$, then all the constraint functions f_t, $t \in T$, are proper. Obviously, both sets $\mathcal{F}(\pi)$ and $\mathcal{F}^{opt}(\pi)$ are closed in X (possibly empty).

It is obvious that the stability of ϑ and $\mathcal{F}^{opt}$ will be greatly influenced by the stability of $\mathcal{F}$, and this is why many results in the present paper rely on suitable adaptations of some others in [17]. This revision of the stability properties, due to the fact that the perturbations of C are measured in a different way in this paper, has been completed with the study of the Lipschitz-like property of $\mathcal{F}$. For the sake of a greater concreteness, in this paper the decision space X is a Banach space (in [17] was a locally convex Hausdorff topological vector space).

1.1. **Antecedents.** Table 1 reviews briefly a non-exhaustive list of relevant works on stability of optimization problems chronologically ordered. Those works dealing with particular types of perturbations, usually right-hand side (RHS) perturbations and/or perturbations which fixed constraint set C, are marked with an asterisk. In most previous works, $C = X$. Abstract minimization problems subject to perturbations can be formulated as

$$\inf \quad f\left(x,\pi\right) \quad \text{s.t. } x \in \mathcal{F}\left(\pi\right),$$

with $\pi \in \Pi$ (the corresponding parameter space), where the implicit constraints determine the stability properties of $\mathcal{F}$ at π, which together with suitable continuity properties of $f\left(\cdot,\pi\right)$ should guarantee the stability of ϑ and $\mathcal{F}^{opt}$ at π. For comparison purpose, we represent here the functional constraints as $f\left(t,\cdot\right) \in K$, where $f\left(t,x\right) := f_t(x)$ and K is a given subset of certain partially ordered space Y (e.g., $Y = \overline{\mathbb{R}}^T$ and $K = \overline{\mathbb{R}}_-^T$ for our problem (P)).

We codify the information in the columns 3-8 of Table 1 as follows:

Col. 3: Banach (Ban), normed (nor), metric (met), locally convex Hausdorff topological vector space (lcH), and topological space (top).

Col. 4: finite (fin), arbitrary (arb), and compact Hausdorff topological space (compH). In case of abstract minimization problems (abstr), there is no explicit information on T, $f(t,\cdot)$, $f(\cdot,x)$, and K.

Col. 5-7: affine (aff), linear (lin), fractional (fract), convex (conv), finite valued (fin), continuous (cont), lower semicontinuous (lsc), upper semicontinuous (usc), arbitrary (arb), and continuously differentiable (diff). In case of abstract minimization problems, no direct information on the constraints is available and the usual allowed perturbations are sequential.

Col. 8: closed (cl) and convex (conv).

For the sake of brevity we do not include in this table information on the parameters space and the only stability concepts considered here are exclusively lower and upper semicontinuity and closedness. This precludes, among other stability concepts related with $\mathcal{F}^{opt}$, the Lipschitzian and Hölder stabilities ([6]), the structural stability ([26], [27]) or the stability of stationary solutions ([22]).

Ref.	Year	X	T	f	$f(t,\cdot)$	$f(\cdot,x)$	K	ϑ	$\mathcal{F}^{opt}$
[23]	1973	top	abstr[1]	lsc or usc	-	-	-	✓	✓
[7]	1982	$\mathbb{R}^n$	top	lin	fract	arb	$\mathbb{R}^T_-$	✓	✓
[4]	1983	met	abstr[2]	lsc or usc	-	-	-	✓	✓
[8]*	1983	$\mathbb{R}^n$	compH	cont	aff	cont	$\mathbb{R}^T_-$		✓
[19]	1983	$\mathbb{R}^n$	compH	lin	aff	cont	$\mathbb{R}^T_-$		✓
[9]	1984	$\mathbb{R}^n$	compH	lin	aff	cont	$\mathbb{R}^T_-$	✓	✓
[15]	1984	norm	compH	cont	aff	cont	$\mathbb{R}^T_-$	✓	
[30]	1985	$\mathbb{R}^n$	fin	fin conv	fin conv/aff	-	$\mathbb{R}^T_-$	✓	✓
[3]*	1997	met	abstr	lsc	-	-	-	✓	
[21]	1998	$\mathbb{R}^n$	arb	lin	aff	arb	$\mathbb{R}^T_-$	✓	✓
[29]	1998	$\mathbb{R}^n$	compH[3]	diff	diff	cont	$\mathbb{R}^T_-$	-	✓
[6]	2000	Ban	arb	cont	cont	-	cl conv	✓	✓
[11]	2001	$\mathbb{R}^n$	arb	lin	aff	arb	$\mathbb{R}^T_-$	✓	✓
[20]	2003	$\mathbb{R}^n$	arb	fin conv	fin conv	arb	$\mathbb{R}^T_-$	✓	✓
[25]	2005	Ban	fin	fin conv	fin conv/aff	-	cl conv	✓	✓[4]
[32]*	2006	met	abstr	fin usc	-	-	-	✓	
[10]*	2007	$\mathbb{R}^n$	met compH	fin conv	fin conv	cont	$\mathbb{R}^T_-$		✓
[18]*	2007	lcH	arb	lsc conv	lsc conv	arb	$\mathbb{R}^T_-$	✓	
[24]*	-	$\mathbb{R}^n$	compH	fin conv	fin conv	cont	$\mathbb{R}^T_-$		✓

Table 1

The closest antecedents of this paper are those works dealing with the stability of $\mathcal{F}$, ϑ, or $\mathcal{F}^{opt}$ for optimization problems as (P) with closed constraint set and lsc constraint functions, whose perturbations are measured by a metric describing the uniform convergence on certain family of sets covering X. [31] analyzes the stability of $\mathcal{F}$ and [20] the corresponding to ϑ and $\mathcal{F}^{opt}$ when $X = \mathbb{R}^n$ and $\Pi_\diamond$ is formed by those parameters $\pi = (f, \{f_t, t \in T;\ C\})$ such that all the involved functions are real-valued and convex, and $C = \mathbb{R}^n$ is invariant under perturbations. [17] deals with the stability of $\mathcal{F}$, but there we measure the perturbations of C as those corresponding to its indicator function. Finally, [24] studies the stability of $\mathcal{F}$ and $\mathcal{F}^{opt}$ when $X = \mathbb{R}^n$ and $\Pi_\diamond$ is formed by those parameters $\pi = (f, \{f_t, t \in T;\ C\})$ such that $C \subset \mathbb{R}^n$ is a fixed closed convex set, T is a

[1]§ 4 is devoted to feasible maps determined by a finite number of inequalities involving functions $f_i(x),\ i = 1, 2, \ldots, m$, which are lsc at the nominal problem.

[2]Feasible maps determined by a possibly infinite number of quasiconvex inequality constraints are considered several times.

[3]T is not fixed, it varies with the parameter, but it is always compact and unformly bounded.

[4]The paper deals with the well posedness of convex programs under linear perturbations of the objective functions and right-hand side perturbations of the constraints.

given compact metric space, all the involved functions are real-valued and convex, and $f_{(\cdot)}(x)$ is continuous on T for all $x \in \mathbb{R}^n$. Under these strong conditions, [24] provides sufficient conditions for the lower semicontinuity and the Lipschitz-like property of $\mathcal{F}^{opt}$. The extension of the latter results to the more general setting is a challenging task to be handled in a forthcoming paper. The results in [24] has been extended in [12] and [13] to vector optimization problems with similar assumptions, analyzing the stability of the Pareto efficient set instead of the optimal set.

1.2. **Organization.** The paper is organized as follows: §2 introduces the basic notation and the stability concepts considered in this paper, §3 introduces a metric on the parameter space Π, under which it is a complete metric space, §4 revises the closedness and lower semicontinuity of $\mathcal{F}$ under the current assumption of this paper, §5 analyzes the Lipschitz-like property of $\mathcal{F}$, §6 is focused on the upper semicontinuity of ϑ, §7 is devoted to the lower semicontinuity of this function and, finally, §8 studies the stability properties, mainly the closedness and upper semicontinuity, of $\mathcal{F}^{opt}$.

2. Preliminaries

The dual space of X is denoted by X^*. $\mathbb{B}$ denotes the closed unit ball in X whereas θ denotes indistinctly the zero of X and of X^*. For a set $D \subset X$, we denote with D^c, $\operatorname{conv} D$, and $\operatorname{cone} D$ the complement of D, the convex hull of D, and the convex conical hull of $D \cup \{\theta\}$, respectively. If $D = \{d_s, s \in S\}$, denoting by $\mathbb{R}^{(S)}$ the linear space of mappings from S to $\mathbb{R}$ with finite support and by $\mathbb{R}_+^{(S)}$ its positive cone, we can write $\operatorname{cone} D = \left\{\sum_{s\in S} \lambda_s d_s : \lambda \in \mathbb{R}_+^{(S)}\right\}$ and $\operatorname{conv} D = \left\{\sum_{s\in S} \lambda_s d_s : \lambda \in \mathbb{R}_+^{(S)}, \sum_{s\in S} \lambda_s = 1\right\}$.

From the topological side, we denote by $\mathcal{N}(x)$ the family of all the neighborhoods of $x \in X$ and by $\operatorname{cl} D$ the closure of D, if $D \subset X$, and the closure of D w.r.t. the weak* topology, if $D \subset X^* \times \mathbb{R}$. The *indicator function* δ_D is defined as $\delta_D(x) = 0$ if $x \in D$, and $\delta_D(x) = +\infty$ if $x \notin D$. D is a nonempty closed convex set if and only if δ_D is a proper lsc convex function.

Now let $h\colon X \to \mathbb{R} \cup \{+\infty\}$. The effective domain, the graph, and the epigraph of h are $\operatorname{dom} h = \{x \in X : h(x) < +\infty\}$, $\operatorname{gph} h = \{(x, \gamma) \in X \times \mathbb{R} : h(x) = \gamma\}$, and $\operatorname{epi} h = \operatorname{gph} h + \operatorname{cone}\{(\theta, 1)\}$, respectively, whereas the conjugate function of h, $h^*\colon X^* \to \mathbb{R} \cup \{\pm\infty\}$, is defined by

$$h^*(v) = \sup\{\langle v, x\rangle - h(x) : x \in \operatorname{dom} h\}.$$

It is well-known that, if h is a proper lsc convex function, then h^* enjoys the same properties and its conjugate, denoted by $h^{**}\colon X \to \mathbb{R} \cup \{\pm\infty\}$, defined by

$$h^{**}(x) = \sup\{\langle v, x\rangle - h^*(v) : v \in \operatorname{dom} h^*\},$$

coincides with h. One observes that δ_C^* is the *support function* of C, whose epigraph $\operatorname{epi}\delta_C^*$ is a closed convex cone.

Let $\sigma = \{f_t(x) \leq 0, t \in T; x \in C\} \in \operatorname{dom}\mathcal{F}$ be a *convex system* (i.e., f_t is convex for all $t \in T$ and C is a convex set), and let $v \in X^*$ and $\alpha \in \mathbb{R}$. Then the asymptotic Farkas' Lemma (Theorem 4.1 in [16]) establishes that

$$f_t(x) \leq 0 \ \ \forall t \in T, \ x \in C \implies \langle v, x \rangle \leq \alpha$$

if and only if

$$(v, \alpha) \ \in \ \operatorname{cl}\operatorname{cone}\left(\bigcup_{t\in T}\operatorname{epi} f_t^* \cup \operatorname{epi}\delta_C^*\right). \tag{2.1}$$

Let $A_1, A_2, \ldots, A_n, ..$ be a sequence of nonempty subsets of a first countable Hausdorff space Y. We consider the set of *limit points* of this sequence

$$y \in \operatorname*{Li}_{n\to\infty} A_n \Leftrightarrow \begin{cases} \text{there exist } y_n \in A_n, \ n = 1, 2, \ldots, \\ \text{such that } (y_n)_{n\in\mathbb{N}} \text{ converges to } y; \end{cases}$$

and the set of *cluster points*

$$y \in \operatorname*{Ls}_{n\to\infty} A_n \Leftrightarrow \begin{cases} \text{there exist } n_1 < n_2 < \cdots < n_k \cdots, \text{ and associated } y_{n_k} \in A_{n_k} \\ \text{such that } (y_{n_k})_{k\in\mathbb{N}} \text{ converges to } y. \end{cases}$$

Clearly $\operatorname{Li}_{n\to\infty} A_n \subset \operatorname{Ls}_{n\to\infty} A_n$ and both sets are closed. We say that $A_1, A_2, \ldots, A_n, \ldots$ is *Kuratowski-Painlevé* convergent to the closed set A if $\operatorname{Li}_{n\to\infty} A_n = \operatorname{Ls}_{n\to\infty} A_n = A$, and we write then $A = K - \lim_{n\to\infty} A_n$.

We recall here some well-known concepts in the theory of multivalued mappings. Let Y and Z be two topological spaces, and consider a set-valued mapping $\mathcal{S}\colon Y \rightrightarrows Z$. We say that $\mathcal{S}$ is *lower semicontinuous* (in the Berge sense) at $y \in Y$ (lsc, in brief) if, for each open set $W \subset Z$ such that $W \cap \mathcal{S}(y) \neq \emptyset$, there exists an open set $V \subset Y$ containing y, such that $W \cap \mathcal{S}(y') \neq \emptyset$ for each $y' \in V$. $\mathcal{S}$ is said to be *lsc* if it is lsc at every point of Y.

The following property (used, e.g., in [5] and [38]) is closely related to the lower semicontinuity of $\mathcal{F}^{opt}$. We say that $\mathcal{S}$ is *uniformly compact-bounded* at $y_0 \in Y$ if there exist a compact set $K \subset Y$ and a neighborhood V of y_0 such that

$$y \in V \Longrightarrow \mathcal{S}(y) \subset K.$$

$\mathcal{S}$ is *upper semicontinuous* (in the Berge sense) at $y \in Y$ (usc, in brief) if, for each open set $W \subset Z$ such that $\mathcal{S}(y) \subset W$, there exists an open set $V \subset Y$ containing y, such that $\mathcal{S}(y') \subset W$ for each $y' \in V$. $\mathcal{S}$ is *usc* if it is usc at every point of Y.

If both Y and Z are first countable Hausdorff spaces, $\mathcal{S}$ is *closed* at $y \in Y$ if for every pair of sequences $(y_n)_{n\in\mathbb{N}} \subset Y$ and $(z_n)_{n\in\mathbb{N}} \subset Z$ satisfying $z_n \in \mathcal{S}(y_n)$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} z_n = z$, one has $z \in \mathcal{S}(y)$. $\mathcal{S}$ is said to be *closed* if it is closed at every $y \in Y$. Obviously, $\mathcal{S}$ is closed if and only if its *graph*, $\operatorname{gph}\mathcal{S} := \{(y, z) \in Y \times Z : z \in \mathcal{S}\,(y)\}$, is a closed set in the product space.

We say that $\pi = (f, \sigma)$ (or, equivalently, σ) satisfies the *strong Slater condition* if there exists some $\bar{x} \in \operatorname{int} C$ and some $\rho > 0$ such that $f_t(\bar{x}) < -\rho$ for all $t \in T$ (i.e., $g(\bar{x}) < -\rho$). In such a case, $\bar{x}$ is called *strong Slater* (SS*) point* of π (or σ) with associated constant ρ.

The SS condition in this paper is stronger than the one introduced in the previous paper [17] as far as $\bar{x}$ is required here to be an element of $\operatorname{int} C$ instead of C itself. The reason is the different type of convergence of sequences in both works.

Observe that π satisfies the SS condition if and only if the optimal value of the following problem (whose structure is similar to that of (P), but with linear objective function),

$$\begin{array}{lll} (\mathrm{P}_{SS}) & \inf & -y \\ & \text{s.t.} & f_t(x) + y \leq 0, \quad \forall t \in T; \\ & & x \in \operatorname{int} C, \;\; y \in \mathbb{R}, \end{array}$$

is negative, in which case it is unnecessary to solve (P_{SS}) until optimality. According to [17, Theorem 5.1], if $C = X$ and σ is convex, then the SS condition is equivalent to $(\theta, 0) \notin \operatorname{cl} \operatorname{conv} \left(\bigcup_{t \in T} \operatorname{epi} f_t^* \right)$ (a condition involving the data).

3. The parameter space

In order to define a suitable topology on the parameter space Π we follow different steps:

1*st*. We start by equipping the space $\mathcal{V}$ of all functions $f \colon X \to \mathbb{R} \cup \{+\infty\}$ which are lsc with the topology of uniform convergence on bounded sets of X. It is well known (see, for instance, [5, p.79] that a compatible metric for this topology is given by

$$d(f, h) := \sum_{k=1}^{+\infty} 2^{-k} \min\{1, \sup_{\|x\| \leq k} |f(x) - h(x)|\}.$$

Here, by convention, we understand that

$$(+\infty) - (+\infty) = 0, \;\; |-\infty| = |+\infty| = \infty.$$

It is worth noting that $d(f, h) = 0$ implies that $|f(x) - h(x)| = 0$ for all $x \in X$. By our convention, either $f(x) = h(x) = +\infty$ or $f(x) = h(x) \in \mathbb{R}$, and $f = h$.

The following lemmas will be very useful in the sequel:

Lemma 1 (Lemma 3.1 in [17])**.** *Let us define*

$$d_k(f, h) := \sup_{\|x\| \leq k} |f(x) - h(x)|, \qquad k = 1, 2, \ldots,$$

and let $k \in \mathbb{N}$ and $\varepsilon > 0$ be given. Then, there exists $\rho > 0$ such that $d_k(f, h) < \varepsilon$ for each pair $f, h \in \mathcal{V}$ satisfying $d(f, h) < \rho$.

Lemma 1 yields the following implication

$$\forall k \in \mathbb{N},\ d(f, f_n) \longrightarrow 0 \implies d_k(f, f_n) \longrightarrow 0.$$

Lemma 2 (Lemma 3.2 in [17]). *For each $\varepsilon > 0$, there exist $k \in \mathbb{N}$ and $\rho > 0$ such that $d(f, h) < \varepsilon$ for each pair $f, h \in \mathcal{V}$ satisfying $d_k(f, h) < \rho$.*

A sequence of extended functions $f_n\colon X \to \mathbb{R} \cup \{+\infty\}$, $n \in \mathbb{N}$, converges uniformly to $f\colon X \to \mathbb{R} \cup \{+\infty\}$ on a set $B \subset X$ when, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in B$ and for all $n \geq n_0$. Recalling the above convention, this is equivalent to assert that $B \cap \operatorname{dom} f_n = B \cap \operatorname{dom} f$ for all $n \geq n_0$ and the restriction of f_n to the latter set converges uniformly (in the sense of finite-valued functions) to the restriction of f_n to the same set.

Proposition 1. *Let $f, f_n \in \mathcal{V}$, $n = 1, 2, \ldots$. Then $d(f_n, f) \to 0$ if and only if the sequence $f_1, f_2, \ldots, f_n, \ldots$ converges uniformly to f on the bounded sets of X.*

Proof. It is immediate consequence of the previous lemmas. ∎

Proposition 2 (Proposition 3.5 in [17]). *$(\mathcal{V}, d)$ is a complete metric space.*

$2nd$. In the space of closed sets in X we shall consider the *Attouch-Wets topology*, which is the inherited topology from the one considered in $\mathcal{V}$ under the identification $C \longleftrightarrow d_C(\cdot)$, with $d_C(x) := \inf_{c \in C} \|x - c\|$ (provided that $C \neq \emptyset$, otherwise $d(x, \emptyset) = +\infty$). This topology is compatible with the distance

$$\widetilde{d}(C, D) := \sum_{k=1}^{+\infty} 2^{-k} \min \left\{ 1,\ \sup_{\|x\| \leq k} |d_C(x) - d_D(x)| \right\}.$$

Observe that $\widetilde{d}(C, D) = d(d_C, d_D)$. The space of all closed sets in X equipped with this distance $\widetilde{d}$ becomes a complete metric space. It is obvious that in this space, the sequence of nonempty closed sets $(C_n)_{n \in \mathbb{N}}$ converges in the sense of Attouch-Wets to the nonempty closed set C if the sequence of functions $(d_{C_n})_{n \in \mathbb{N}}$ converges to d_C uniformly on the bounded sets of X. Thanks to the fact that X is Banach, we can apply Lemma 3.1.1 in [5] to guarantee that if the sequence $(d_{C_n})_{n \in \mathbb{N}}$ converges uniformly on bounded sets of X to a continuous function h, there exists a nonempty closed set C such that $h = d_C$.

Moreover, Corollary 3.1.8 in [5] establishes that the sequence of nonempty closed sets $(C_n)_{n \in \mathbb{N}}$ converges in the sense of Attouch-Wets to the nonempty closed C if and only if

$$\forall k \in \mathbb{N}\colon \lim_{n \to \infty} \max \{ e(C_n \cap k\mathbb{B}, C), e(C \cap k\mathbb{B}, C_n) \} = 0, \tag{3.1}$$

where

$$e(A, B) := \sup_{a \in A} d_B(a) = \inf\{\alpha > 0 :\ B + \alpha\mathbb{B} \supset A\}.$$

Remark 1. *In* [17] *we used another distance between sets. More precisely, in that paper we considered as the distance between two closed sets C and D the distance $d(\delta_C, \delta_D)$, i.e. the distance between their indicator functions. In fact, the topology associated with $\widetilde{d}$ is coarser. Actually, $d(\delta_C, \delta_{C_n}) \to 0$ if and only if for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $C \cap k\mathbb{B} = C_n \cap k\mathbb{B}$ for every $n \geq n_k$, but this entails $e(C_n \cap k\mathbb{B}, C) = e(C \cap k\mathbb{B}, C_n) = 0$ for every $n \geq n_k$, implying trivially (3.1), i.e. $\widetilde{d}(C, C_n) \to 0$.*

It is said that the sequence of functions $(f_n)_{n\in\mathbb{N}} \in \mathcal{V}$, converges to $f \in \mathcal{V}$ in the *sense of Attouch-Wets* if

$$\lim_{n\to\infty} (\operatorname{epi} f_n) = \operatorname{epi} f$$

for the topology of Attouch-Wets in $X \times \mathbb{R}$ equipped with the box norm $\|(x, \alpha)\| = \max\{\|x\|, |\alpha|\}$.

Lemma 7.1.2 in [5] shows that if we consider $f, f_n \in \mathcal{V},\ n = 1, 2, \ldots$, such that $d(f_n, f) \to 0$, and f is real valued, then $(f_n)_{n\in\mathbb{N}}$ converges to f in the *sense of Attouch-Wets.*

Finally, we consider the parameter spaces of constraint systems

$$\Theta := \left\{ \sigma = \{f_t, t \in T;\ C\} : \begin{array}{ll} f_t \in \mathcal{V}, & \forall t \in T, \\ C \text{ closed} & \end{array} \right\},$$

and optimization problems $\Pi = \mathcal{V} \times \Theta$. We consider Θ equipped with the metric $\mathbf{d}$ such that

$$\mathbf{d}(\sigma, \sigma') := \max\{\sup_{t\in T} d(f_t, f_t'),\ \widetilde{d}(C, C')\},$$

for $\sigma = \{f_t, t \in T; C\}),\ \sigma' = \{f_t', t \in T; C'\} \in \Theta$, with the convention that $\sup_{t\in T} d(f_t, f_t') = 0$ whenever $T = \emptyset$. In order to get the product topology on $\Pi = \mathcal{V} \times \Theta$, we define

$$\mathbf{d}(\pi, \pi') := \max\{d(f, f'), \mathbf{d}(\sigma, \sigma')\} \tag{3.2}$$

for any pair $\pi = (f, \sigma)$, $\pi' = (f', \sigma') \in \Pi$ (for simplicity, we use the same notation for the metrics on Θ and Π).

Proposition 3. $(\Pi, \mathbf{d})$ *is a complete metric space.*

Proof. The proof can be adapted from the proof of Proposition 3.7 in [17]. To show that $(\Pi, \mathbf{d})$ is a metric space is also here a straightforward consequence that d and $\widetilde{d}$ are metrics (see [17, p.2265]).

Now we prove that $(\Pi, \mathbf{d})$ is complete. Let $\pi_n = (f^n, \{f_t^n, t \in T; C_n\})$, $n = 1, 2, \ldots$, be a Cauchy sequence in $(\Pi, \mathbf{d})$, and we deal with the more complicated case, i.e., when $T \neq \emptyset$. We must prove that there is a system $\pi \in \Pi$ such that $\mathbf{d}(\pi_n, \pi) \to 0$ as n tends to infinity.

Let $\varepsilon \in]0,1[$ be fixed. For any $k \in \mathbb{N}$, by Lemma 1, there is $\rho_k > 0$ such that

$$d(f,h) < \rho_k \implies d_k(f,h) < \varepsilon. \tag{3.3}$$

As $(\pi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists $n_0 > 0$ such that for any $m, n \geq n_0$, one has

$$\mathbf{d}(\pi_n, \pi_m) = \max\{d(f^n, f^m), \sup_{t\in T} d(f_t^n, f_t^m), d(d_{C_n}, d_{C_m})\} < \rho_k,$$

which gives, for all $m, n \geq n_0$,

$$\begin{aligned} d(f^n, f^m) &< \rho_k, \\ d(f_t^n, f_t^m) &< \rho_k, \quad \forall t \in T, \\ d(d_{C_n}, d_{C_m}) &< \rho_k. \end{aligned}$$

It follows from (3.3) that, for all $m, n \geq n_0$ and for all x such that $\|x\| \leq k$,

$$\begin{aligned} |f^n(x) - f^m(x)| &< \varepsilon, \\ |f_t^n(x) - f_t^m(x)| &< \varepsilon, \quad \forall t \in T, \\ |d_{C_n}(x) - d_{C_m}(x)| &< \varepsilon. \end{aligned} \tag{3.4}$$

By a reasoning similar to the one used in the proof of Proposition 3.5 in [17], and applying also Lemma 3.1.1 in [5], one concludes the existence of functions f and f_t, $t \in T$, belonging all to $\mathcal{V}$, and a nonempty closed C such that as $n \to \infty$

$$d(f^n, f) \to 0, \ d(f_t^n, f_t) \to 0 \text{ for all } t \in T, \text{ and } \widetilde{d}(C_n, C) \to 0.$$

Let $\pi := (f, \{f_t, t \in T; C\})$. We now prove that $\mathbf{d}(\pi_n, \pi) \to 0$ as $n \to \infty$.

With $\varepsilon > 0$ fixed, by Lemma 2, there exist k_0 and $\rho_0 > 0$ such that for any $f, h \in \mathcal{V}$,

$$d_{k_0}(f,h) < \rho_0 \implies d(f,h) < \varepsilon. \tag{3.5}$$

Without loss of generality we can take $\rho_0 < 1$. Since $(\pi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists $n_1 > 0$ such that for all $n, m \geq n_1$,

$$\mathbf{d}(\pi_n, \pi_m) = \max\{d(f^n, f^m), \sup_{t\in T} d(f_t^n, f_t^m), d(d_{C_n}, d_{C_m})\} < \rho_0 2^{-k_0-1}.$$

This yields, thanks to the fact that $\rho_0 < 1$,

$$\begin{aligned} d_{k_0}(f^n, f^m) &< \rho_0/2, \\ d_{k_0}(f_t^n, f_t^m) &< \rho_0/2, \quad \forall t \in T, \\ d_{k_0}(d_{C_n}, d_{C_m}) &< \rho_0/2, \end{aligned}$$

which, in turn, implies that (letting $m \to \infty$)

$$\begin{aligned} d_{k_0}(f^n, f) &< \rho_0, \\ d_{k_0}(f_t^n, f_t) &< \rho_0, \quad \forall t \in T, \\ d_{k_0}(d_{C_n}, d_C) &< \rho_0. \end{aligned}$$

By (3.5), the last inequalities yield respectively for all $n \geq n_1$:

$$d(f^n, f) < \varepsilon, \ \sup_{t\in T} d(f_t^n, f_t) \leq \varepsilon, \ \text{and} \ d(d_{C_n}, d_C) < \varepsilon.$$

Therefore, $\mathbf{d}(\pi_n, \pi) \to 0$ as n tends to ∞. Consequently, $(\Pi, \mathbf{d})$ is complete. ∎

4. Lower semicontinuity of the feasible set mapping revisited

The closedness of the feasible set mapping $\mathcal{F}$ is established in the following proposition. It is a consequence of the fact that the topology considered in Θ gives rise to the uniform convergence on bounded sets.

Proposition 4. *The feasible set mapping $\mathcal{F}$ is closed on Θ.*

Proof. The proof follows from the same line of reasoning as in the proof of [17, Theorem 4.1], but applying the suitable changes concerning the treatment of the distance of $\widetilde{d}$ (as it is done in the proof of Proposition 3). ∎

In order to revisit the property of lower semicontinuity of $\mathcal{F}$ in the new scenario considered in this paper, we need some previous technical lemmas.

Lemma 3. *Let C be a closed set in X, $x_0 \in \operatorname{int} C$, and consider $\varepsilon > 0$ such that $x_0 + \varepsilon\mathbb{B} \subset C$. Then there is $\rho > 0$ such that, for any closed set $C' \subset X$,*

$$\widetilde{d}(C, C') < \rho \implies (x_0 + \varepsilon\mathbb{B}) \cap C' \neq \emptyset.$$

Proof. Take a positive integer k such that

$$x_0 + \varepsilon\mathbb{B} \subset k\mathbb{B}.$$

Given ε and k, apply Lemma 1 to conclude the existence of $\rho > 0$ such that

$$\widetilde{d}(C, C') < \rho \implies \sup_{\|x\|\leq k} |d_C(x) - d_{C'}(x)| < \varepsilon.$$

Therefore,

$$z \in x_0 + \varepsilon\mathbb{B} \implies d_C(z) = 0 \text{ and } d_{C'}(z) < \varepsilon. \tag{4.1}$$

Now, if $(x_0 + \varepsilon\mathbb{B}) \cap C' = \emptyset$, we have $d_{C'}(x_0) \geq \varepsilon$, and this contradicts (4.1). ∎

Lemma 4. *Consider $\sigma = \{f_t, t \in T; C\} \in \Theta$ and suppose that the marginal function $g = \sup_{t\in T} f_t$ is usc. If $\hat{x}$ is an SS-point of σ, then there exists $\varepsilon > 0$ such that*

$$\left.\begin{array}{l} x \in \hat{x} + \varepsilon\mathbb{B} \\ \mathbf{d}(\sigma, \sigma') < \varepsilon \end{array}\right\} \implies g'(x) < 0,$$

with $\sigma' = \{f'_t, t \in T; C'\} \in \Theta$ and $g' := \sup_{t\in T} f'_t$.

Proof. Let $\hat{x}$ be an SS-point of π. There exists $\rho > 0$ such that $g(\hat{x}) \leq -\rho$. Take ρ_1 and ρ_2 such that $0 < \rho_2 < \rho_1 < \rho$. Since g is usc there must exist ε_1 such that

$$x \in \hat{x} + \varepsilon_1 \mathbb{B} \implies g(x) \leq -\rho_1.$$

Let k be an integer satisfying $\hat{x} + \varepsilon_1 \mathbb{B} \subset k\mathbb{B}$. Lemma 1 applies to ensure the existence of $\varepsilon_2 > 0$ such that

$$\mathbf{d}(\sigma, \sigma') < \varepsilon_2 \implies \sup_{\|x\| \leq k} |f_t(x) - f_t'(x)| < \rho_1 - \rho_2, \quad \forall t \in T.$$

Let us set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Combining the previous arguments, if $x \in \hat{x} + \varepsilon \mathbb{B}$ and $\mathbf{d}(\sigma, \sigma') < \varepsilon$, we get

$$\begin{aligned} g'(x) &= g(x) + g'(x) - g(x) = g(x) + \sup_{t \in T} f_t'(x) - \sup_{t \in T} f_t(x) \\ &\leq g(x) + \sup_{t \in T} [f_t'(x) - f_t(x)] \leq -\rho_1 - \rho_2 + \rho_1 = -\rho_2 < 0, \end{aligned}$$

which is desired. ∎

We now consider a convex set C with $\operatorname{int} C \neq \emptyset$. Without any loss of generality by translation, we assume that $\theta \in \operatorname{int} C$ and consider the *Minkovski gauge function* defined as

$$p_C(x) := \inf\{\lambda \geq 0 \mid x \in \lambda C\},$$

and, for any positive real number $\mu \in [0, 1[$, define a set

$$C_\mu := \{x \in X \mid p_C(x) \leq \mu\}.$$

It is worth observing (see [38]) that p_C is a continuous sublinear function, and hence, C_μ is a closed and convex set such that $C_\mu \subset \operatorname{int} C$ by the accesibility lemma. The latter inclusion becomes an equation when $C = X$ for all $\mu \in [0, 1[$. The following proposition shows that one can adjust μ to get C_μ arbitrarily close to C (in the sense that $\widetilde{d}(C, C_\mu)$ is arbitrarily small).

Lemma 5. *Let C be a closed convex set in X such that* $\operatorname{int} C \neq \emptyset$. *Given $\varepsilon > 0$, there exists $\mu \in]0, 1[$ such that*

$$\widetilde{d}(C, C_\mu) \leq \varepsilon.$$

Proof. We assume w.l.o.g. that $\theta \in \operatorname{int} C$ and let us fix $\varepsilon > 0$. From Lemma 2 we know that for this given ε, there exist $k \in \mathbb{N}$ and $\rho > 0$ such that

$$\left.\begin{array}{l} \left|d_{C_\mu}(x) - d_C(x)\right| \leq \rho \\ \forall x \text{ such that } \|x\| \leq k \end{array}\right\} \implies \widetilde{d}(C, C_\mu) \leq \varepsilon. \tag{4.2}$$

We now show that there does exist a $\mu > 0$ that satisfies the antecedent of (4.2).

We can suppose that $\rho < 1$ and take any μ satisfying

$$\mu \in \left[1 - \frac{\rho}{2k}, 1\right[.$$

Pick any $x_0 \in C \cap 2k\mathbb{B}$ and represent it as
$$x_0 = \mu x_0 + (1-\mu)x_0 = \mu x_0 + (1-\mu)2k\frac{x_0}{2k}.$$
Since $\mu x_0 \in \mu C$, one has $p_C(\mu x_0) \leq \mu$ which entails $\mu x_0 \in C_\mu$, and so,
$$x_0 \in C_\mu + \rho\mathbb{B}.$$
Thanks to the arbitrariness of x_0, this proves that
$$C \cap 2k\mathbb{B} \ \subset \ C_\mu + \rho\mathbb{B}.$$
We now apply Lemma 4.34 (c) in [36] (still valid for normed spaces) to conclude that
$$d_{C_\mu} \leq d_C + \rho \ \text{ on } \ k\mathbb{B}.$$
This together with the obvious inequality $d_{C_\mu} \geq d_C$ yields
$$|d_{C_\mu}(x) - d_C(x)| \leq \rho$$
for all x satisfying $\|x\| \leq k$. The conclusion now follows from (4.2). ∎

The necessary and sufficient conditions for the lower semicontinuity of the feasible set mapping $\mathcal{F}$ are given in the next result. Remember that the system σ is said to be *Tuy regular* if there exists $\varepsilon > 0$ such that for any $u \in \mathbb{R}^T$ and for any nonempty convex set $C' \subset X$ satisfying $\max\{\sup_{t\in T}|u_t|, \ \widetilde{d}(C,C')\} < \varepsilon$, the system $\sigma' = \{f_t(x) - u_t \leq 0, t \in T; \ x \in C'\} \in \operatorname{dom}\mathcal{F}$. The last definition is inspired in a similar one of H. Tuy ([37]).

Theorem 1. *Let $\mathcal{F}\colon \Theta \rightrightarrows X$ and $\sigma = \{f_t, t \in T; C\} \in \Theta$. If $T = \emptyset$, then $\mathcal{F}$ is lsc whenever C is convex and* $\operatorname{int} C \neq \emptyset$*. Otherwise, consider the following statements associated with $\sigma \in \operatorname{dom}\mathcal{F}$:*
(i) $\mathcal{F}$ is lsc at σ;
(ii) $\sigma \in \operatorname{int}\operatorname{dom}\mathcal{F}$;
(iii) σ is Tuy regular;
(iv) σ satisfies the strong Slater condition;
(v) $\mathcal{F}(\sigma)$ is the closure of the set of SS points of σ.

Then, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (iv). Moreover, if C is convex, and $\operatorname{int} C \neq \emptyset$*, then (i) $\Rightarrow$ (v) and (iii) $\Rightarrow$ (iv).*

If, in addition, the functions f_t, $t \in T$, are convex and the corresponding marginal function $g = \sup_{t\in T} f_t$ is usc, then all the statements (i) $-$ (v) are equivalent.

Proof. First we consider $T = \emptyset$. Let W be an open set in X such that $W \cap C \neq \emptyset$. Then $W \cap \operatorname{int} C \neq \emptyset$ because $C = \operatorname{cl}\operatorname{int} C$. Let $x_0 \in X$ and $\varepsilon > 0$ be such that $x_0 + \varepsilon\mathbb{B} \subset W \cap \operatorname{int} C$. By Lemma 3, there exists $\rho > 0$ such that, for all closed set C',
$$\widetilde{d}(C,C') < \rho \implies (x_0 + \varepsilon\mathbb{B}) \cap C' \neq \emptyset.$$
So, $W \cap C' \neq \emptyset$ for all $\sigma' \in \Theta$ such that $\mathbf{d}(\sigma,\sigma') < \rho$.

Now we assume that $T \neq \emptyset$. The proofs of the implications $(i) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (iii)$ are the same as in [17, Theorem 5.1].

$(v) \Rightarrow (iv)$ is trivial because if we denote by $\mathcal{F}_{SS}(\sigma)$ the set of SS points of σ, $\sigma \in \operatorname{dom} \mathcal{F}$ entails $\emptyset \neq \mathcal{F}(\sigma) = \operatorname{cl} \mathcal{F}_{SS}(\sigma)$ and $\mathcal{F}_{SS}(\sigma)$ cannot be empty.

Assume that C is convex and $\operatorname{int} C \neq \emptyset$ and we will show that $(iii) \Longrightarrow (iv)$. The argument is again the same as in the corresponding part in [17, Theorem 5.1] except the choice of C' for σ'. Indeed, if σ is Tuy regular then, for some $\varepsilon > 0$, the system $\sigma' := \{f_t - w_t, t \in T; C'\} \in \Theta$ is consistent whenever

$$\max\left\{\sup_{t\in T} |w_t|;\ \widetilde{d}(C, C')\right\} < \varepsilon. \tag{4.3}$$

It now follows from Lemma 5 that there is $\mu \in]0,1[$ such that $\widetilde{d}(C, C_\mu) \leq \varepsilon$. Note that here, since C is convex, and $\mu < 1$, we have $\emptyset \neq C_\mu \subset \operatorname{int} C$ (see [38], page 4, and the proof of Lemma 5).

Let $w_t = -\frac{\varepsilon}{2}$ for all $t \in T$ and $C' = C_\mu$. Then $\sigma' \in \Theta$, and (4.3) holds, and hence, σ' is consistent, i.e., $\mathcal{F}(\sigma') \neq \emptyset$. It is obvious that any $\tilde{x} \in \mathcal{F}(\sigma')$ is an SS-point of σ, so that (iv) holds.

Also under the assumptions that C is convex, and $\operatorname{int} C \neq \emptyset$, let us prove that $(i) \Longrightarrow (v)$. It is evident that $\operatorname{cl} \mathcal{F}_{SS}(\sigma) \subset \mathcal{F}(\sigma)$ because $\mathcal{F}_{SS}(\sigma) \subset \mathcal{F}(\sigma)$ and $\mathcal{F}(\sigma)$ is closed. Reasoning by contradiction, let us suppose that there exists $x^1 \in \mathcal{F}(\sigma) \diagdown \operatorname{cl} \mathcal{F}_{SS}(\sigma)$, and take an open set W such that $x^1 \in W$ and $W \cap \mathcal{F}_{SS}(\sigma) = \emptyset$. Since $x^1 \in W \cap \mathcal{F}(\sigma)$, (i) states the existence of $\varepsilon > 0$ such that $W \cap \mathcal{F}(\sigma') \neq \emptyset$ for every $\sigma' \in \Theta$ such that $\mathbf{d}(\sigma, \sigma') < \varepsilon$.

Take now $\sigma' = \{f'_t, t \in T; C'\}$ such that $f'_t := f_t + \frac{\varepsilon}{2}$, for all $t \in T$, and $C' = C_\mu$ with

$$C_\mu := z + \{x \in X :\ p_{C-z}(x) \leq \mu\},$$

where $z \in \operatorname{int} C$ and $\mu \in]0,1[$ satisfies

$$\widetilde{d}(C, C_\mu) < \varepsilon$$

(see Lemma 5). Since $\mathbf{d}(\sigma, \sigma') < \varepsilon$, one has $W \cap \mathcal{F}(\sigma') \neq \emptyset$, but $\mathcal{F}(\sigma') \subset \mathcal{F}_{SS}(\sigma)$, contradicting $W \cap \mathcal{F}_{SS}(\sigma) = \emptyset$.

Finally we prove that $(iv) \Rightarrow (i)$ assuming that the functions $f_t,\ t \in T$, are convex, C is convex with nonempty interior, and the marginal function g is usc. Let $\tilde{x}$ be an SS-point of σ. Consider an open set W in X such that

$$W \cap \mathcal{F}(\sigma) \neq \emptyset,$$

and take $\bar{x} \in W \cap \mathcal{F}(\sigma)$. By the convexity of $\mathcal{F}(\sigma)$ (namely, f_t and C are convex), the point

$$\hat{x} = (1-\lambda)\tilde{x} + \lambda\bar{x} \in W \cap \mathcal{F}(\sigma)$$

if $\lambda \in [0,1[$ is large enough. Moreover, since $\tilde{x} \in \operatorname{int} C$ and $\bar{x} \in C$, $\hat{x} \in \operatorname{int} C$ is an SS-point of σ too for all $\lambda \in [0,1[$.

According to Lemma 4 there will exist $\varepsilon > 0$ such that

$$\left.\begin{array}{r} x \in \hat{x} + \varepsilon\mathbb{B} \\ \mathbf{d}(\sigma, \sigma') < \varepsilon \end{array}\right\} \Longrightarrow g'(x) < 0. \tag{4.4}$$

We shall take ε small enough to guarantee that

$$\hat{x} + \varepsilon\mathbb{B} \ \subset \ C \cap W.$$

Apply now Lemma 3 to get $\rho_1 > 0$ such that, for all closed $C' \subset X$,

$$\widetilde{d}(C, C') < \rho_1 \implies (\hat{x} + \varepsilon\mathbb{B}) \cap C' \neq \emptyset,$$

and define $\rho := \min\{\rho_1, \varepsilon\}$. Now, if $\sigma' \in \Theta$, $\mathbf{d}(\sigma, \sigma') < \rho$ and $x^1 \in (\hat{x} + \varepsilon\mathbb{B}) \cap C'$, we have $g'(x^1) < 0$ by (4.4), and $x^1 \in C' \cap W$. So $x^1 \in W \cap \mathcal{F}(\sigma')$, and this set is non-empty, entailing the lower semicontinuity of $\mathcal{F}$ at σ. ∎

Remark 2. *The argument of Theorem 1 remains valid when Θ is replaced by some subspace $\Theta_\diamond$ such that $\sigma = \{f_t, t \in T; C\} \in \Theta_\diamond$ entails that $\sigma' := \{f_t + w_t, t \in T; C_\mu\} \in \Theta_\diamond$ for all $w \in \mathbb{R}^T$ and $\mu \in [0, 1[$. Families of functions and sets satisfying this condition are the lsc functions on X whose local minima are global minima (or the lsc convex functions, or the continuous convex functions, or the continuous affine functions) together with the closed convex subsets of X (or the singleton family $\{X\}$).*

We are obtaining straightforward consequences from Theorem 1 (actually from its argument), and from the next results, for two particular subsets of Θ. We denote by Θ_1 the set of parameters of the form $\{f_t, t \in T; C\}$ such that f_t is a continuous affine functional, for all $t \in T$, and $C = X$, and by Θ_2 the set of parameters such that f_t is a usc convex function, for all $t \in T$, and $C = X$. Obviously, $\Theta_1 \subset \Theta_2$. The next corollary of Theorem 1 is also straightforward consequence of [17, Theorem 5.1] and [31, Theorem 4.1].

Corollary 1. *Let $|T| < \infty$, and $\mathcal{F}colon\Theta_i \rightrightarrows X$ with $i = 1, 2$. Then the statements (i)-(v) in Theorem 1 are equivalent to each other.*

5. Lipschitz-like property of the feasible set mapping

This section deals with the Lipschitz-like property of $\mathcal{F}$ (or, equivalently, with the metric regularity of $\mathcal{F}^{-1}$). It is well-known that this property has important consequences in the overall stability of a system σ, as well as in the sensitivity analysis of perturbed systems, affecting even the numerical complexity of the algorithms conceived for finding a solution of the system. Many authors ([1], [2], [14], [28], [29], [30], [33], [34], [35], [39], etc.) investigated this property and explored the relationship of this property with standard constraint qualifications as Mangasarian-Fromovitz CQ, Slater CQ, Robinson CQ, etc. For instance, in [29] the relationships among the Lipschitz-like property, Lipschitz-like property with respect to right-hand side (RHS) perturbations, and the extended Mangasarian-Fromowitz CQ are established in a non-convex differentiable setting (see Table 1). In that paper, the authors make use of one result in [14] showing that, under mild conditions, Lipschitz-like property and Lipschitz-like property respect to RHS perturbations are equivalent. Let us remember the definition of Lipschitz-like property applied to our specific mapping:

Definition 1. *$\mathcal{F}$ is said to be* Lipschitz-like at $(\sigma, x) \in \operatorname{gph}\mathcal{F}$ *if there exist real numbers $\varepsilon, \delta > 0$ and $\kappa \geq 0$ such that*

$$\left.\begin{array}{l} \mathbf{d}(\sigma,\sigma') < \delta \\ \|x - x'\| < \varepsilon \end{array}\right\} \Longrightarrow d(x', \mathcal{F}(\sigma')) \leq \kappa \mathbf{d}(\sigma', \mathcal{F}^{-1}(x')). \tag{5.1}$$

(5.1) means that the distance $d(x', \mathcal{F}(\sigma'))$ is bounded from above by $\kappa\mathbf{d}(\sigma', \mathcal{F}^{-1}(x'))$, and this is specially useful if the residual $\mathbf{d}(\sigma', \mathcal{F}^{-1}(x'))$ can be easily computed.

The existence of a constraint set C makes the computation of $\mathbf{d}(\sigma', \mathcal{F}^{-1}(x'))$ very difficult, and this is why the Lipschitz-like property is useless in this case. Nevertheless, when we assume that C is the whole space X, the property makes sense, and it is strongly related to other stability properties already studied in the previous section. In fact, if C is constantly equal to X and $\sigma' = \{f_t', t \in T\}$, it is straightforward that

$$\mathbf{d}(\sigma', \mathcal{F}^{-1}(x')) = \left[\sup_{t\in T} f_t'(x')\right]_+ \equiv [g'(x')]_+\,, \tag{5.2}$$

where $g' = \sup_{t\in T} f_t'$ and $[\alpha]_+ := \max\{\alpha, 0\}$.

Observe that for a system having a constraint set, say $\sigma' = \{f_t', t \in T, C'\}$, we have

$$\mathbf{d}(\sigma', \mathcal{F}^{-1}(x')) = \max\left\{[g'(x')]_+\,, \widetilde{d}(C', \mathcal{C}_{x'}(X))\right\}, \tag{5.3}$$

where $\mathcal{C}_{x'}(X)$ is the family of all the closed convex sets $C \subset X$ such that $x' \in C$, and

$$\widetilde{d}(C', \mathcal{C}_{x'}(X)) = \inf\left\{\widetilde{d}(C', C) :\ C \in \mathcal{C}_{x'}(X))\right\}.$$

It is obvious that this residual (5.3) is far from being easily computable.

Since $C = X$ throughout this section, we can write $\sigma = \{f_t, t \in T\}$ instead of $\sigma = \{f_t, t \in T; X\}$.

Theorem 2. *Let $\mathcal{F}\colon \Theta_\diamond \rightrightarrows X$ and $(x, \sigma) \in \operatorname{gph}\mathcal{F}^{-1}$ with $\sigma = \{f_t, t \in T\}$. Then the following statements are true:*

(i) Let $\Theta_\diamond$ be the set of parameters whose constraint set is X. If f_t is convex for all $t \in T$, $g = \sup_{t\in T} f_t$ is usc at x, and $\mathcal{F}$ is Lipschitz-like at (σ, x), then $\mathcal{F}$ is lsc at σ.

(ii) Let $\Theta_\diamond$ be the set of parameters whose constraint functions are convex and whose constraint set is X. If X is a Hilbert space, and $\mathcal{F}$ is lsc at σ, then $\mathcal{F}$ is Lipschitz-like at (σ, x).

Proof. The constraint set in the systems considered here is X, so that we can just write $\sigma' = \{f_t'(x) \leq 0, t \in T\}$ for all $\sigma' \in \Theta_\diamond$.

(i) We are assuming the existence of real numbers $\varepsilon, \delta > 0$ and $\kappa \geq 0$ such that (5.1) holds. By taking $x' = x$ in (5.1) and since $\sigma \in \mathcal{F}^{-1}(x)$, we conclude that

$$d(x, \mathcal{F}(\sigma')) \leq \kappa \mathbf{d}(\sigma', \mathcal{F}^{-1}(x)) \leq \kappa\delta,$$

and $d(x, \mathcal{F}(\sigma'))$ is finite, entailing $\mathcal{F}(\sigma') \neq \emptyset$ provided that $\mathbf{d}(\sigma, \sigma') < \delta_2$. In other words, σ is in the interior of the effective domain of $\mathcal{F}$ when it is restricted to those systems in Θ for which $C = X$. The conclusion follows from Remark 2.

(ii) We are assuming that X is a Hilbert space, $\mathcal{F}$ is lsc at σ, and that $x \in \mathcal{F}(\sigma)$.

Take an arbitrary fixed $\varepsilon > 0$. Since $\mathcal{F}(\sigma)$ is the closure of the set of strong Slater points by Remark 2, there must exist $\widehat{y} \in x + \varepsilon\mathbb{B}$ and $\rho > 0$ such that $g(\widehat{y}) \equiv \sup_{t\in T} f_t(\widehat{y}) \leq -\rho$. A standard argument yields the existence of a positive scalar δ that $g'(\widehat{y}) \equiv \sup_{t\in T} f_t'(\widehat{y}) \leq -\rho/2$ if $\mathbf{d}(\sigma, \sigma') < \delta$ and $\sigma' = \{f_t', t \in T\}$, with f_t', $t \in T$, being convex. Observe that $\widehat{y} \in \mathcal{F}(\sigma')$ and so, $(x+\varepsilon\mathbb{B}) \cap \mathcal{F}(\sigma') \neq \emptyset$.

Now we take an arbitrary $x' \in x + \varepsilon\mathbb{B}$ and $\sigma' = \{f_t', t \in T\} \in \Theta$ such that $f_t', t \in T$, are convex, and $\mathbf{d}(\sigma, \sigma') < \delta$.

Since we are in a Hilbert space, there will exist a point $y_{\sigma'} \in \mathcal{F}(\sigma')$ such that $d(x', \mathcal{F}(\sigma')) = \|x' - y_{\sigma'}\|$, and this point is characterized by the inequality $\langle x' - y_{\sigma'}, y - y_{\sigma'}\rangle \leq 0$ for all $y \in \mathcal{F}(\sigma')$. We shall analyze only the nontrivial case $x' \notin \mathcal{F}(\sigma')$.

Now, $\langle x' - y_{\sigma'}, y\rangle \leq \langle x' - y_{\sigma'}, y_{\sigma'}\rangle$ is a consequent relation of the system σ', and we apply the Farkas' Lemma to conclude, by (2.1), that

$$(x' - y_{\sigma'}, \langle x' - y_{\sigma'}, y_{\sigma'}\rangle) \ \in \ \operatorname{cl}\operatorname{cone}\left(\bigcup_{t\in T} \operatorname{epi}(f_t')^*\right).$$

Then there exist nets $\{\lambda^\alpha\}_{\alpha\in\Delta} \subset \mathbb{R}_+^{(T)}$, $\{u_t^\alpha\}_{\alpha\in\Delta} \subset \operatorname{dom}(f_t')^*$, $t \in T$, and $\{\beta^\alpha\}_{\alpha\in\Delta} \subset \mathbb{R}_+$, such that

$$\begin{aligned} \lim\nolimits_\alpha \textstyle\sum_{t\in T} \lambda_t^\alpha u_t^\alpha = x' - y_{\sigma'}, \\ \lim\nolimits_\alpha \left(\textstyle\sum_{t\in T} \lambda_t^\alpha (f_t')^*(u_t^\alpha) + \beta^\alpha\right) = \langle x' - y_{\sigma'}, y_{\sigma'}\rangle. \end{aligned} \tag{5.4}$$

Therefore, from (5.4) we get

$$\lim_\alpha \left\{\sum_{t\in T} \lambda_t^\alpha [\langle u_t^\alpha, x'\rangle - (f_t')^*(u_t^\alpha)] - \beta^\alpha\right\} = \|x' - y_{\sigma'}\|^2. \tag{5.5}$$

Since for each $\alpha \in \Delta$ and $t \in T$,

$$\langle u_t^\alpha, x'\rangle - (f_t')^*(u_t^\alpha) \leq (f_t')^{**}(x') = f_t'(x'),$$

from (5.5) we derive

$$\left\| x' - y_{\sigma'} \right\|^2 \leq \overline{\lambda} \sup_{t \in T} f'_t(x'), \tag{5.6}$$

where $\bar{\lambda} := \limsup_\alpha \sum_{t\in T} \lambda_t^\alpha$, $\bar{\lambda} \in \mathbb{R} \cup \{+\infty\}$.

From (5.4) we also obtain

$$\langle x' - y_{\sigma'}, \widehat{y} - y_{\sigma'} \rangle = \lim_\alpha \left\{ \sum_{t\in T} \lambda_t^\alpha \left[\langle u_t^\alpha, \widehat{y} \rangle - (f'_t)^*(u_t^\alpha) \right] - \beta^\alpha \right\} \leq -\overline{\lambda} \frac{\rho}{2},$$

which gives rise to

$$\overline{\lambda} \leq \frac{2}{\rho} \left\| x' - y_{\sigma'} \right\| \left\| \widehat{y} - y_{\sigma'} \right\|,$$

that together with (5.6) yields

$$d(x', \mathcal{F}(\sigma')) = \left\| x' - y_{\sigma'} \right\| \leq \frac{2}{\rho} \left\| \widehat{y} - y_{\sigma'} \right\| \sup_{t\in T} f'_t(x'). \tag{5.7}$$

Moreover

$$\begin{aligned} \left\| \widehat{y} - y_{\sigma'} \right\| &\leq \left\| \widehat{y} - x' \right\| + \left\| x' - y_{\sigma'} \right\| \\ &\leq 2 \left\| \widehat{y} - x' \right\| \leq 4\varepsilon. \end{aligned} \tag{5.8}$$

Combining (5.2), (5.7), and (5.8), we conclude that (5.1) is satisfied with $\kappa = \frac{8\varepsilon}{\rho}$. ∎

The next example shows that Theorem 2 fails when the convexity assumption on the constraint functions f_t, $t \in T$, is replaced by the weaker one that the local minima of the marginal function g are global minima (under which Theorem 1 remains valid according to [17, Theorem 5.1] provided that $C = X$).

Example 1. *Let $X = \mathbb{R}$, $T = \{1\}$, $f_1(x) = -x^2$, and $\sigma = \{f_1(x) \leq 0\}$. Let $W \neq \emptyset$ be an arbitrary open set in $\mathbb{R}$ and take $z \in W$ and $k \in \mathbb{N}$ such that $|z| \leq k$. Let $\sigma' = \{f'_1(x) \leq 0\} \in \Theta$ be such that $\mathbf{d}(\sigma', \sigma) = d(f'_1, f_1) < 2^{-k} z^2$. Then, $2^{-k} \left| f'_1(z) + z^2 \right| < 2^{-k} z^2$, so that $f'_1(z) < 0$ and $z \in \mathcal{F}(\sigma') \cap W$. Hence $\mathcal{F}$ is lsc at σ. Now we assume that $\mathcal{F}$ is Lipschitz-like at $(\sigma, 0)$. Let $\varepsilon, \delta > 0$ and $\kappa \geq 0$ be such that (5.1) holds. Let $\sigma_n = \{f_1^n(x) \leq 0\}$, with $f_1^n(x) = f + \frac{1}{n}$, and $x_n = 0$, $n \in \mathbb{N}$. Then, for sufficient large n, we must have $d(x_n, \mathcal{F}(\sigma_n)) \leq \kappa \mathbf{d}(\sigma_n, \mathcal{F}^{-1}(x_n))$, i.e., $\frac{1}{\sqrt{n}} \leq \frac{\kappa}{n}$. Multiplying by n both members of the latter inequality and taking limits as $n \to \infty$ we get a contradiction. Hence $\mathcal{F}$ is not Lipschitz-like at $(\sigma, 0)$ and statement (ii) in Theorem 2 does not hold.*

Corollary 2. *Let $\mathcal{F} \colon \Theta_i \rightrightarrows X$ with $i = 1, 2$ and $(\sigma, x) \in \operatorname{gph} \mathcal{F}$. If $\mathcal{F}$ is lsc at σ and X is a Hilbert space, then $\mathcal{F}$ is Lipschitz-like at (σ, x). Conversely, if $\mathcal{F}$ is Lipschitz-like at (σ, x) and $|T| < \infty$, then $\mathcal{F}$ is lsc at σ.*

6. Upper semicontinuity of the optimal value function

We now study the upper semicontinuity of the optimal value function ϑ.

Theorem 3. *Let $\pi = (f, \sigma) \in \Pi$. The following statements hold:*
(i) If $\mathcal{F}$ is lsc at σ then ϑ is usc at π provided that f is usc.
(ii) If ϑ is usc at π then $\mathcal{F}$ is lsc at σ provided that the functions $f_t,\ t \in T$, are convex, C is convex, $\operatorname{int} C \neq \emptyset$, and the corresponding marginal function $g = \sup_{t\in T} f_t$ is usc.

Proof. (i) Assume that $\mathcal{F}$ is lsc at σ and f is usc. We need to prove that if μ is a real number such that $\vartheta(\pi) < \mu$, there exists a neighborhood U of π such that

$$\vartheta(\pi') \leq \mu, \quad \forall \pi' = (f', \sigma') \in U.$$

Since $\vartheta(\pi) < \mu$, there exists $x_0 \in \mathcal{F}(\sigma)$ such that

$$f(x_0) < \mu.$$

Consider a natural number k such that $x_0 \in k\mathbb{B}$. Then $V_1 := (k+1)\mathbb{B}$ is a neighborhood of x_0.

Set $\varepsilon := \frac{1}{2}(\mu - f(x_0))$. We now can apply Lemma 1 to conclude the existence of $\rho > 0$ such that

$$d(f, f') < \rho \implies d_{k+1}(f, f') < \varepsilon.$$

In other words,

$$d(f, f') < \rho \implies \sup_{x\in V_1} |f(x) - f'(x)| < \varepsilon. \tag{6.1}$$

Since f is usc at x_0 there must exist V_2, a neighborhood of x_0, such that

$$f(x) \leq f(x_0) + \varepsilon, \quad \forall x \in V_2. \tag{6.2}$$

If $x \in V := V_1 \cap V_2$ and $d(f, f') < \rho$, we have from (6.1) and (6.2)

$$f'(x) < f(x) + \varepsilon \leq f(x_0) + 2\varepsilon = \mu. \tag{6.3}$$

Since $\mathcal{F}$ is lsc at σ by assumption, and $\mathcal{F}(\sigma) \cap V \neq \emptyset$ (it contains x_0), there is a neighborhood of σ, W, such that

$$\sigma' \in W \implies \mathcal{F}(\sigma') \cap V \neq \emptyset.$$

Consider $U := \{f' \in \mathcal{V} :\ d(f, f') < \rho\} \times W$, that is a neighborhood of π. Given $\pi' = (f', \sigma') \in U$ we can select an arbitrary $x^1 \in \mathcal{F}(\sigma') \cap V$. Then by (6.3), we conclude

$$f'(x^1) < \mu,$$

and hence, $\vartheta(\pi') < \mu$.

(ii) Assume that ϑ is usc at $\pi \in \Pi$, and that the functions $f_t,\ t \in T$, are convex, C is convex, $\operatorname{int} C \neq \emptyset$, and the corresponding marginal function $g = \sup_{t\in T} f_t$ is usc. This ensures that, for any fixed μ with $\vartheta(\pi) < \mu$, there exists $\rho > 0$ such that

$$\vartheta(\pi') < \mu$$

whenever $\pi' \in \Pi$, $d(\pi, \pi') < \rho$. This particularly means that for these $\pi' = (f', \sigma')$, $\mathcal{F}(\sigma') \neq \emptyset$. In other words, $\pi \in \operatorname{int} \operatorname{dom} \mathcal{F}$. The conclusion now follows from Theorem 1. ∎

Corollary 3. *Let $|T| < \infty$, $\Pi_i = \Theta_i \times \mathcal{V}$ with $i = 1, 2$, $\mathcal{F} \colon \Theta_i \rightrightarrows X$, $\vartheta \colon \Pi_i \rightrightarrows \overline{\mathbb{R}}$, and $\pi = (f, \sigma) \in \Pi_i$. If $\mathcal{F}$ is lsc at σ and f is usc, then ϑ is usc at π. Conversely, if ϑ is usc at π and $|T| < \infty$, then $\mathcal{F}$ is lsc at σ.*

7. Lower semicontinuity of the optimal value function

We shall consider from now on the so-called *sublevel sets mapping* $\mathcal{L}$: $\Pi \times \mathbb{R} \rightrightarrows X$ defined as follows:

$$\mathcal{L}(\pi, \lambda) := \{x \in \mathcal{F}(\sigma) : \ f(x) \leq \lambda\}, \text{ with } \pi = (f, \sigma).$$

Obviously, if $\lambda < \vartheta(\pi)$ trivially $\mathcal{L}(\pi, \lambda) = \emptyset$. Moreover, $\mathcal{L}(\pi, \vartheta(\pi)) = \mathcal{F}^{opt}(\pi)$.

Theorem 4. *The mapping $\mathcal{L}$ is closed.*

Proof. We have to prove the closedness of $\mathcal{L}$ at any $(\pi, \lambda) \in \Pi \times \mathbb{R}$ such that $\mathcal{L}(\pi, \lambda)$ is nonempty. To this aim take a sequence $(\pi_k, \lambda_k) \in \Pi \times \mathbb{R}$, $\pi_k = (f^k, \sigma_k)$, $k = 1, 2, \ldots$, such that $\lim_{k\to\infty}(\pi_k, \lambda_k) = (\pi, \lambda) = ((f, \sigma), \lambda)$, and a sequence $x_k \in \mathcal{L}(\pi_k, \lambda_k)$, $k = 1, 2, \ldots$, such that $\lim_{k\to\infty} x_k = x$.

Since $\mathcal{F}$ is closed on Θ by Proposition 4, and $x_k \in \mathcal{F}(\sigma_k)$, $k = 1, 2, \ldots$, we get $x \in \mathcal{F}(\sigma)$. In addition, if $k_0 \in \mathbb{N}$ is big enough to satisfy $(x_k)_{k\in\mathbb{N}} \subset k_0\mathbb{B}$, we can write

$$\begin{aligned} f(x_k) &= f^k(x_k) + f(x_k) - f^k(x_k) \\ &\leq f^k(x_k) + d_{k_0}(f, f^k) \\ &\leq \lambda_k + d_{k_0}(f, f^k). \end{aligned}$$

The lower semicontinuity of f leads us to

$$f(x) \leq \liminf_{k\to\infty} f(x_k) \leq \lim_{k\to\infty} (\lambda_k + d_{k_0}(f, f^k)) = \lambda,$$

and so $x \in \mathcal{L}(\pi, \lambda)$. ∎

Theorem 5. *If $\mathcal{L}$ is uniformly compact-bounded at $(\pi, \vartheta(\pi)) \in \Pi \times \mathbb{R}$, then ϑ is lsc at π.*

Proof. Since $\mathcal{L}$ is uniformly compact-bounded at $(\pi, \vartheta(\pi))$, if we take a convenient $\lambda > \vartheta(\pi)$, we have that $\mathcal{L}(\pi, \lambda)$ is compact. Then Theorem 2.6 in [6] applies to conclude that $\mathcal{F}^{opt}(\pi) = \mathcal{L}(\pi, \vartheta(\pi))$ is a nonempty compact set.

Given $\varepsilon > 0$ we shall prove the existence of $\delta > 0$ such that

$$d(\pi, \pi') < \delta \Longrightarrow \vartheta(\pi') \geq \vartheta(\pi) - \varepsilon. \tag{7.1}$$

Since $\mathcal{L}$ is uniformly compact-bounded at $(\pi, \vartheta(\pi))$, Proposition 6.3.2 and Theorem 6.1.16 in [5], together with Theorem 4, apply to conclude that $\mathcal{L}$ is usc at $(\pi, \vartheta(\pi))$.

Take now the open set

$$W := \left\{ x \in X : \ f(x) > \vartheta(\pi) - \frac{\varepsilon}{2} \right\}.$$

Obviously $\mathcal{L}(\pi, \vartheta(\pi)) \subset W$. The upper semicontinuity of $\mathcal{L}$ at $(\pi, \vartheta(\pi))$ entails the existence of $\delta > 0$ such that

$$d(\pi, \pi') < \delta, \ |\vartheta(\pi) - \lambda'| < \delta \Longrightarrow \mathcal{L}(\pi', \lambda') \subset W.$$

We shall take δ small enough to guarantee that actually we have

$$d(\pi, \pi') < \delta, \ |\vartheta(\pi) - \lambda'| < \delta \Longrightarrow \mathcal{L}(\pi', \lambda') \subset W \cap K, \tag{7.2}$$

K being a compact set.

Now we choose in (7.2) $\lambda' = \vartheta(\pi)$ and π' satisfying $d(\pi, \pi') < \delta$. Two cases may arise:

a) If $\mathcal{L}(\pi', \vartheta(\pi)) = \emptyset$, then $\vartheta(\pi') \geq \vartheta(\pi) > \vartheta(\pi) - \varepsilon$ (possibly, $\vartheta(\pi') = +\infty$).

b) If $\mathcal{L}(\pi', \vartheta(\pi)) \neq \emptyset$, then Theorem 2.6 in [6] and (7.2) provide

$$\emptyset \neq \mathcal{F}^{opt}(\pi') \subset \mathcal{L}(\pi', \vartheta(\pi)) \subset W \cap K. \tag{7.3}$$

Pick now $x_0 \in \mathcal{F}^{opt}(\pi')$ and $k \in \mathbb{N}$ such that $\|x_0\| \leq k$. Again by Lemma 1, if δ is small enough, we can be sure that

$$\|x\| \leq k \Longrightarrow |f'(x) - f(x)| \leq \frac{\varepsilon}{2}. \tag{7.4}$$

Combining (7.3) and (7.4), and recalling the definition of W, we get

$$\begin{aligned} \vartheta(\pi') &= f'(x_0) = f(x_0) + f'(x_0) - f(x_0) \\ &> \vartheta(\pi) - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = \vartheta(\pi) - \varepsilon. \end{aligned}$$

Hence (7.1) holds. ∎

Theorem 6. *Consider $\pi = (f, \sigma) = (f, \{f_t, t \in T; C\}) \in \Pi$ with $X = \mathbb{R}^n$. Suppose that the functions $f, f_t, t \in T$, are convex and that C is convex. If $\mathcal{F}^{opt}(\pi)$ is a nonempty compact set, then $\mathcal{L}$ is uniformly compact-bounded at $(\pi, \vartheta(\pi))$.*

Proof. Let us consider the system in $\mathbb{R}^n$

$$\widetilde{\sigma} := \{f_t(x) \leq 0, \ t \in T; \ f(x) - \vartheta(\pi) \leq 0; \ d_C(x) \leq 0\}.$$

Obviously, we can represent $\widetilde{\sigma}$ as follows

$$\widetilde{\sigma} := \left\{ f_t(x) \leq 0, \ t \in \widetilde{T}; \ d_C(x) \leq 0 \right\},$$

with index set $\widetilde{T} := T \cup \{t_0\}, \ t_0 \notin T$, and $f_{t_0}(x) := f(x) - \vartheta(\pi)$.

It is evident that, if we represent by $\widetilde{\mathcal{F}}$ the feasible set mapping for convex systems having $\widetilde{T}$ as index set, aside the constraint $x \in C \Leftrightarrow d_C(x) \leq 0$

(C is closed), we have $\widetilde{\mathcal{F}}(\widetilde{\sigma}) = \mathcal{F}^{opt}(\pi)$. Then, the assumption of compactness and non-emptyness of this set implies that $\widetilde{\mathcal{F}}$ is usc at $\widetilde{\sigma}$, according to [17, Proposition 7.5].

Now let us define

$$W := \mathcal{F}^{opt}(\pi) + \{x \in \mathbb{R}^n : \|x\| < 1\} \text{ and } K := \mathcal{F}^{opt}(\pi) + \mathbb{B}.$$

For the extended index set, we shall consider perturbed systems of $\widetilde{\sigma}$ of the form

$$\widetilde{\sigma}' = \{f_t'(x) \leq 0,\ t \in T;\ f'(x) - \lambda' \leq 0;\ d_{C'}(x) \leq 0\}.$$

The upper semicontinuity of $\widetilde{\mathcal{F}}$ at $\widetilde{\sigma}$ entails the existence of $\varepsilon > 0$ such that

$$\widetilde{\mathbf{d}}(\widetilde{\sigma}, \widetilde{\sigma}') < \varepsilon \implies \widetilde{\mathcal{F}}(\widetilde{\sigma}') \subset W, \tag{7.5}$$

where

$$\begin{aligned}\widetilde{\mathbf{d}}(\widetilde{\sigma}, \widetilde{\sigma}') &= \max\{\sup_{t \in \widetilde{T}} d(f_t, f_t'), d(d_C, d_{C'})\} \\ &= \max\{d(f - \vartheta(\pi), f' - \lambda'), \sup_{t \in T} d(f_t, f_t'), d(d_C, d_{C'})\}.\end{aligned}$$

Let us take the problem $\pi' = (f', \sigma') = (f', \{f_t', t \in T; C'\})$ with $f',\ f_t', t \in T$, convex and C' convex, such that $\mathbf{d}(\pi, \pi') < \varepsilon/2$, and the scalar λ' such that $|\vartheta(\pi) - \lambda'| < \varepsilon/2$. Then

$$d(f - \vartheta(\pi), f' - \lambda') \leq d(f, f') + |\vartheta(\pi) - \lambda'| < \varepsilon,$$

implying $\widetilde{\mathbf{d}}(\widetilde{\sigma}, \widetilde{\sigma}') < \varepsilon$. Now (7.5) gives rise to the following implication

$$\left.\begin{array}{l}\mathbf{d}(\pi, \pi') < \varepsilon/2 \\ |\vartheta(\pi) - \lambda'| < \varepsilon/2\end{array}\right\} \implies \widetilde{\mathcal{F}}(\widetilde{\sigma}') = \mathcal{L}(\pi', \lambda') \subset K,$$

and this means that $\mathcal{L}$ is included in the compact set K around $(\pi, \vartheta(\pi))$. ∎

The next result extends [20, Theorem 3.1 (ii)] to convex semi-infinite problems with extended constraint functions and a constraint set (here the involved functions are not necessarily finite-valued and the constraint set is not necessarily the whole space).

Corollary 4. *Consider $\pi = (f, \sigma) \in \Pi$ with $X = \mathbb{R}^n$. Suppose that the functions $f, f_t, t \in T$, are convex and that C is convex. If $\mathcal{F}^{opt}(\pi)$ is a nonempty compact set, then ϑ is lsc at π.*

Proof. It is a straightforward consequence of Theorems 5 and 6. ∎

The next example shows that Theorem 6 and Corollary 4 fail when the convexity assumption on the objective f and the constraint functions f_t, $t \in T$, is replaced by the weaker one that all the local minima of f and $g = \sup_{t \in T} f_t$ are global minima of f and g, respectively, even though $C = X = \mathbb{R}^n$.

Example 2. *Let* $X = \mathbb{R}$ *and* $\pi = (f, \sigma) \in \Pi,$ *where* $f(x) = \min\left\{x^2, \frac{1}{|x|+1}\right\}$ *and* σ *is as in Example 1.*

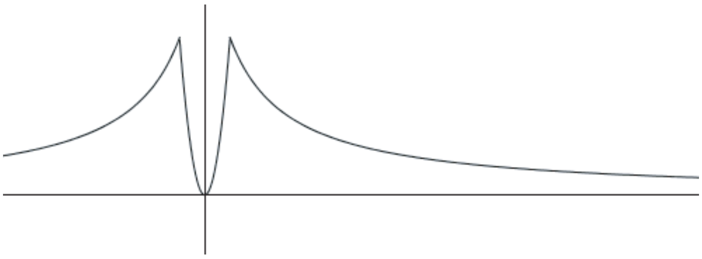

FIGURE 1. gph f

We have $\mathcal{F}^{opt}(\pi) = \{0\}$ *and* $\vartheta(\pi) = 0$. *Consider the sequence* $(\pi_n)_{n\in\mathbb{N}}$ *such that* $\pi_n = (f^n, \sigma)$ *and* f^n *is the result of replacing the branch of* gph f *on* $[n, +\infty[$ *by the union of the segment* $\left[\left(n, \frac{1}{n+1}\right), \left(\frac{n^2+10n+5}{(n+1)^2}, -\frac{1}{4}\right)\right]$ *(tangent to* gph f *at* $(n, f(n))$*) with the half line* $\left(\frac{n^2+10n+5}{(n+1)^2}, -\frac{1}{4}\right) + \mathbb{R}_+(1,0)$. *In other words,*

$$f^n(x) = \begin{cases} f(x), & \text{if } x < n, \\ \frac{-x+2n+1}{(n+1)^2}, & \text{if } n \le x \le \frac{n^2+10n+5}{(n+1)^2}, \\ -1, & \text{otherwise.} \end{cases}$$

Then $d(f^n, f) \le 2^{1-n}$ *for all* $n \in \mathbb{N}$ *and so* $\mathbf{d}(\pi_n, \pi) = d(f^n, f) \to 0$. *Since*

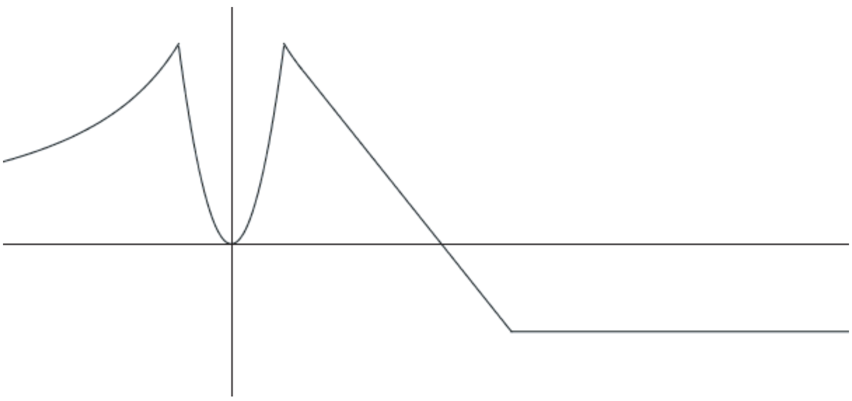

FIGURE 2. gph f^1

$\mathcal{L}(\pi_n, 0) = \{0\} \cup [2n+1, +\infty[$ *is unbounded for all* $n \in \mathbb{N}$, $\mathcal{L}$ *is not uniformly compact-bounded at* $(\pi, 0)$. *Moreover,* $\vartheta(\pi) > -1$ *whereas* $\vartheta(\pi_n) = -\frac{1}{4} < -1$ *for all* $n \in \mathbb{N}$, *so that* ϑ *is not lsc at* π.

8. Stability analysis of the optimal set mapping

This section starts with a sufficient condition for the closedness of $\mathcal{F}^{opt}$.

Theorem 7. *Consider $\pi = (f, \sigma) \in \Pi$ such that f is usc and $\mathcal{F}$ is lsc at σ. Then $\mathcal{F}^{opt}$ is closed at π.*

Proof. We shall analyze the nontrivial case $\mathcal{F}^{opt}(\pi) \neq \emptyset$. Take a sequence $\pi_k = (f^k, \sigma_k) \in \Pi$, $k = 1, 2, \ldots$, such that $\lim_{k\to\infty} \pi_k = \pi$, and a sequence $x_k \in \mathcal{F}^{opt}(\pi_k)$, $k = 1, 2, \ldots$, such that $\lim_{k\to\infty} x_k = x$. We shall prove that $x \in \mathcal{F}^{opt}(\pi)$.

Since $\mathcal{F}$ is closed on Θ by Proposition 4, and $x_k \in \mathcal{F}(\sigma_k)$, $k = 1, 2, \ldots$, we get $x \in \mathcal{F}(\sigma)$. Moreover if $k_0 \in \mathbb{N}$ is big enough to satisfy $(x_k)_{k\in\mathbb{N}} \subset k_0\mathbb{B}$, we write this time

$$\begin{aligned} f(x_k) &= f^k(x_k) + f(x_k) - f^k(x_k) \\ &\leq \vartheta(\pi_k) + d_{k_0}(f, f^k). \end{aligned}$$

Taking limits:

$$f(x) = \lim_{k\to\infty} f(x_k) \leq \liminf_{k\to\infty} \vartheta(\pi_k). \tag{8.1}$$

But thanks to Theorem 3(i) one gets

$$\limsup_{k\to\infty} \vartheta(\pi_k) \leq \vartheta(\pi). \tag{8.2}$$

From (8.1) and (8.2) finally we derive

$$f(x) \leq \vartheta(\pi),$$

and $x \in \mathcal{F}^{opt}(\pi)$. ∎

Theorem 8. *Consider $\pi = (f, \sigma) \in \Pi$ such that f is usc, $\mathcal{F}$ is lsc at σ, and $\mathcal{L}$ is uniformly compact-bounded at $(\pi, \vartheta(\pi)) \in \Pi \times \mathbb{R}$. Then, ϑ is continuous at π and $\mathcal{F}^{opt}$ is usc at π.*

Proof. The assumption on $\mathcal{L}$ guarantees that ϑ is finite-valued in some neighborhood of π. The continuity of ϑ comes from Theorem 3 and Theorem 5. Moreover $\mathcal{F}^{opt}$ is closed at π by Theorem 7. Let us see that $\mathcal{F}^{opt}$ is also uniformly compact-valued at π.

Because $\mathcal{L}$ is uniformly compact-bounded at $(\pi, \vartheta(\pi))$ there exist a compact set K in X and a $\delta > 0$ such that

$$\mathbf{d}(\pi, \pi') < \delta,\ |\vartheta(\pi) - \lambda'| < \delta \Longrightarrow \mathcal{L}(\pi', \lambda') \subset K. \tag{8.3}$$

The continuity of ϑ entails the existence of $\delta_1 > 0$ such that

$$\mathbf{d}(\pi, \pi') < \delta_1 \Longrightarrow |\vartheta(\pi) - \vartheta(\pi')| < \delta.$$

Thus, if $\mathbf{d}(\pi, \pi') \leq \min\{\delta, \delta_1\}$ we have $\mathbf{d}(\pi, \pi') < \delta$ and $|\vartheta(\pi) - \vartheta(\pi')| < \delta$. By (8.3)

$$\mathcal{L}(\pi', \vartheta(\pi')) = \mathcal{F}^{opt}(\pi') \subset K.$$

Now we conclude that $\mathcal{F}^{opt}$ is usc at π by applying again Lemma 6.3.2 in [5] because $\mathcal{F}^{opt}$ is closed and uniformly compact-bounded at π. ∎

Corollary 5. *Consider $\pi = (f, \sigma) \in \Pi$ with $X = \mathbb{R}^n$. Suppose that the functions $f, f_t, t \in T$, are convex, f is in addition usc, and that C is convex. If $\mathcal{F}^{opt}(\pi)$ is a nonempty compact set and $\mathcal{F}$ is lsc at σ, then ϑ is continuous at π and $\mathcal{F}^{opt}$ is usc at π.*

Proof. It is a straightforward consequence of Theorem 6 and Theorem 8. It is also a consequence of Theorem 4.3.3 in [4]. ∎

Example 2 shows once again that the convexity assumption on the objective function f and the constraint functions f_t, $t \in T$, cannot be replaced by the weaker one that all the local minima of f and $g = \sup_{t \in T} f_t$ are global minima of f and g, respectively, even though $C = X = \mathbb{R}^n$. Indeed, $\mathcal{F}^{opt}(\pi) = \{0\}$ is obviously compact and $\mathcal{F}$ is lsc at σ (recall Example 1), but $\mathcal{F}^{opt}$ is not usc at π (consider a bounded neighborhood of 0 and observe that $\mathcal{F}^{opt}(\pi_n)$ is unbounded for all $n \in \mathbb{N}$).

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